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NOTE ON THE CONGRUENCE $2^{2n} \equiv (-)^n (2n)! / (n!)^2$, WHERE $2n + 1$ IS A PRIME.

By PROF. F. MORLEY, Haverford, Pa.

1. There are two ways of integrating $\cos^{2n+1} x dx$, one by a Fourier series, where we express $\cos^{2n+1} x$ in terms of cosines of multiples of x , the other by a "formula of reduction." Taking the integral from $x = 0$ to $x = \pi/2$, and equating the two forms of the integral, we have an algebraic identity, which will, on a suitable supposition as to the integer n , yield a theorem as to prime numbers, trivial or otherwise. Doing this, we have, first,

$$\begin{aligned} 2^{2n} \cos^{2n+1} x &= \cos(2n+1)x + (2n+1) \cos(2n-1)x \\ &\quad + \frac{(2n+1) \cdot 2n}{1 \cdot 2} \cos(2n-3)x + \dots + \frac{(2n+1)2n \dots (n+2)}{n!} \cos x, \\ 2^{2n} \int_0^{\pi/2} \cos^{2n+1} x dx &= \frac{\sin(2n+1)x}{2n+1} + \frac{2n+1}{2n-1} \sin(2n-1)x + \dots, \\ 2^{2n} \int_0^{\pi/2} \cos^{2n+1} x dx &= (-)^n \left[\frac{1}{2n+1} - \frac{2n+1}{2n-1} + \dots \right]; \end{aligned}$$

and second, from the formula of reduction,

$$\int_0^{\pi/2} \cos^{2n+1} x dx = \frac{2n(2n-2) \dots 2}{(2n+1)(2n-1) \dots 3}; \quad (1)$$

so that the algebraic identity is

$$\begin{aligned} &2^{2n} \frac{2n(2n-2) \dots 2}{(2n+1)(2n-1) \dots 3} \\ &= (-)^n \left[\frac{1}{2n+1} - \frac{2n+1}{2n-1} + \frac{(2n+1)2n}{1 \cdot 2(2n-3)} - \dots + (-)^n \frac{(2n+1)2n \dots (n+2)}{n!} \right]. \end{aligned}$$

Let $2n+1$ be a prime p . Let us use the notation $a/b \equiv 0, \text{ mod } c$, where a/b is a *fraction*, to mean that when the fraction is in its lowest terms the numerator has the factor c . Then, from the above identity,

$$2^{2n} \frac{2n(2n-2) \dots 2}{(2n+1)(2n-1) \dots 3} - (-)^n \frac{1}{2n+1} \equiv 0, \text{ mod } p,$$

or

$$2^{2n} \frac{2n(2n-2) \dots 2}{(2n-1)(2n-3) \dots 1} - (-)^n \equiv 0, \text{ mod } p^2,$$

or

$$2^{4n} \frac{(n!)^2}{(2n)!} - (-)^n \equiv 0, \text{ mod } p^2,$$

or

$$2^{4n} - (-)^n \frac{(2n)!}{(n!)^2} \equiv 0, \text{ mod } p^2, \quad (2)$$

the left hand member being of course an integer.

This result is given in Mathews, Theory of Numbers, p. 318, Ex. 16.

When $n = 1, 2, 3$, the left hand member of (2) is respectively 18, 250, 4116, that is $2 \cdot 3^2$, $2 \cdot 5^3$, $2^2 \cdot 3 \cdot 7^3$. Thus, when $p = 5$ and 7, the left hand member $\equiv 0, \text{ mod } p^3$, not merely $\text{mod } p^2$. I have to prove that this is so when p is a prime > 3 .

2. It is convenient to prove, first, that when $p > 3$

$$\frac{1}{1^2} + \frac{1}{3^2} + \dots + \frac{1}{(p-2)^2} \equiv 0, \text{ mod } p. \quad (3)$$

Using the notation and results of Chrystal, Algebra, Vol. ii, p. 525, let

$$(x+1)(x+2)\dots(x+p-1) = x^{p-1} + A_1 x^{p-2} + \dots + A_{p-1}.$$

We have

$$[(p-1)!]^2 \left[\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{(p-1)^2} \right] = A_{p-2}^2 - 2A_{p-1}A_{p-3};$$

or since $A_{p-2}, A_{p-3} \equiv 0 \text{ mod } p$, if $p > 3$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{(p-1)^2} \equiv 0 \text{ mod } p. \quad (4)$$

Now (writing always $2n+1 = p$),

$$\begin{aligned} & \frac{1}{1^2} - \frac{1}{(2n)^2} + \frac{1}{2^2} - \frac{1}{(2n-1)^2} + \dots + \frac{1}{n^2} - \frac{1}{(n+1)^2} \\ &= \frac{(2n)^2 - 1^2}{1^2(2n)^2} + \frac{(2n-1)^2 - 2^2}{2^2(2n-1)^2} + \dots + \frac{(n+1)^2 - n^2}{n^2(n+1)^2} \equiv 0 \text{ mod } p, \end{aligned}$$

for each numerator has, and each denominator has not, the factor $2n+1$.

Hence, both sum and difference of the expressions

$$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{n^2} \quad \text{and} \quad \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2}$$

$\equiv 0, \text{ mod } p$; hence, also, each expression $\equiv 0 \text{ mod } p$. Hence, again,

$$\frac{1}{2^2} + \frac{1}{4^2} + \dots + \frac{1}{(2n)^2} \equiv 0, \text{ mod } p.$$

Hence we can remove the even terms from the statement (4), and there remains (3).

3. Instead of expressing $\cos^p x$ as a sum of cosines, let us take the formula which expresses $\cos px$ as a power-series in $\cos x$. This is shown in treatises on trigonometry to be

$$\begin{aligned} (-)^n \cos px &= p \cos x - \frac{p(p^2-1^2)}{3!} \cos^3 x + \frac{p(p^2-1^2)(p^2-3^2)}{5!} \cos^5 x - \dots \\ &\quad + (-)^n 2^{p-1} \cos^p x. \end{aligned}$$

Multiply by dx and integrate from 0 to $\frac{1}{2}\pi$; then, using (1),

$$\begin{aligned} \frac{1}{p} &= p - \frac{p(p^2-1^2)}{3!} \frac{2}{3} + \frac{p(p^2-1^2)(p^2-3^2)}{5!} \frac{2 \cdot 4}{3 \cdot 5} + \dots \\ &\quad + (-)^n 2^{p-1} \frac{2 \cdot 4 \dots (p-1)}{3 \cdot 5 \dots p}. \end{aligned}$$

Therefore,

$$\frac{1}{p} - (-)^n 2^{p-1} \frac{2 \cdot 4 \dots (p-1)}{3 \cdot 5 \dots p} \equiv p \left\{ 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots + \frac{1}{(p-2)^2} \right\}, \text{ mod } p^3$$

$$\text{i. e. (Art. 2)} \quad \equiv 0, \text{ mod } p^2.$$

Therefore

$$(-)^n 2^{p-1} \frac{2 \cdot 4 \dots (p-1)}{1 \cdot 3 \dots (p-2)} - 1 \equiv 0, \text{ mod } p^3$$

whence, as in Art. 1,

$$2^{4n} - (-)^n \frac{(2n)!}{(n!)^2} \equiv 0, \text{ mod } p^3,$$

where $2n + 1$ is a prime p , greater than 3.